

GEOMETRIC PROPERTIES OF UPPER LEVEL SETS OF LELONG NUMBERS ON PROJECTIVE SPACES

DAN COMAN AND TUYEN TRUNG TRUONG

ABSTRACT. Let T be a positive closed current of unit mass on the complex projective space \mathbb{P}^n . For certain values $\alpha < 1$, we prove geometric properties of the set of points in \mathbb{P}^n where the Lelong number of T exceeds α . We also consider the case of positive closed currents of bidimension $(1,1)$ on multiprojective spaces.

1. INTRODUCTION

Let T be a positive closed current of bidimension (p,p) on a complex manifold M . For $\alpha \geq 0$, we consider the upper level sets of the Lelong numbers $\nu(T, z)$ of T ,

$$\begin{aligned} E_\alpha(T) &= E_\alpha(T, M) = \{z \in M : \nu(T, z) \geq \alpha\}, \\ E_\alpha^+(T) &= E_\alpha^+(T, M) = \{z \in M : \nu(T, z) > \alpha\}. \end{aligned}$$

By [11], if $\alpha > 0$ then $E_\alpha(T)$ is an analytic subvariety of M of dimension at most p , hence $E_0^+(T)$ is an at most countable union of analytic subvarieties of M of dimension $\leq p$.

If $M = \mathbb{P}^n$ we denote by $\|T\|$ the mass (or the degree) of T computed with respect to the Fubini-Study form ω_n on \mathbb{P}^n ,

$$\|T\| = \int_{\mathbb{P}^n} T \wedge \omega_n^p.$$

It is well known that $\nu(T, z) \leq \|T\|$ for every $z \in \mathbb{P}^n$ (see e.g. [2]). Assume without loss of generality that $\|T\| = 1$. When $p = 1$, it was shown in [1, Theorem 1.1] that $E_{2/3}^+(T, \mathbb{P}^n)$ is contained in a (complex) line, while $E_{1/2}^+(T, \mathbb{P}^n)$ is either contained in a line or else it is a finite set such that $|E_{1/2}^+(T, \mathbb{P}^n) \setminus L| = 1$ for some line L . In dimension two, it was proved in [1, Theorem 1.2] that $E_{2/5}^+(T, \mathbb{P}^2)$ is either contained in a conic or else it is a finite set such that $|E_{2/5}^+(T, \mathbb{P}^2) \setminus C| = 1$ for some conic C . When $p = n - 1$, i.e. when T has bidegree $(1, 1)$, it was shown in [2, Proposition 2.2] that $E_{n/(n+1)}^+(T, \mathbb{P}^n)$ is contained in a hyperplane of \mathbb{P}^n . Moreover, these values of α are sharp with respect to these geometric properties.

In this paper we generalize these results to the case of currents of arbitrary bidimension on \mathbb{P}^n . Namely, we prove the following theorems.

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Theorem 1.1. *If T is a positive closed current of bidimension (p, p) on \mathbb{P}^n , $0 < p < n$, with $\|T\| = 1$, then the set $E_{(p+1)/(p+2)}^+(T, \mathbb{P}^n)$ is contained in a p -dimensional linear subspace of \mathbb{P}^n .*

Decreasing the value of α to $p/(p+1)$ we show that $E_{p/(p+1)}^+(T, \mathbb{P}^n)$ has all but at most one point contained in a p -dimensional linear subspace of \mathbb{P}^n . More precisely, the following holds:

Theorem 1.2. *If T is a positive closed current of bidimension (p, p) on \mathbb{P}^n , $0 < p < n$, with $\|T\| = 1$, then the set $E_{p/(p+1)}^+(T, \mathbb{P}^n)$ is either contained in a p -dimensional linear subspace of \mathbb{P}^n or else it is a finite set and $|E_{p/(p+1)}^+(T, \mathbb{P}^n) \setminus L| = p$ for some line L .*

Theorems 1.1 and 1.2 are proved in Section 2. We also prove there another property of the set $E_{p/(p+1)}^+(T, \mathbb{P}^n)$ (see Proposition 2.5), and we give examples of currents showing that the values of α in Theorems 1.1 and 1.2 are sharp with respect to the corresponding geometric property.

Decreasing the value of α further to $(3p-1)/(3p+2)$, we obtain a result analogous to Theorem 1.2 in [1]:

Theorem 1.3. *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^n such that $1 < p < n$, $\|T\| = 1$, and the set $E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n)$ is not contained in a p -dimensional linear subspace of \mathbb{P}^n . If $W = \text{Span}(E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n))$ then $\dim W = p+1$ and there exist plane conics $C_j \subset W$ and points $z_j \in W$, $1 \leq j \leq N_p$, where $N_p = \binom{p+2}{3}$, such that z_j lies in the plane containing C_j and*

$$E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n) \subset C_1 \cup \dots \cup C_{N_p} \cup \{z_1, \dots, z_{N_p}\}.$$

Theorem 1.3 is proved in Section 3. We note there that the corresponding statement does not hold for currents of bidimension $(1, 1)$. However, we give in Theorem 3.3 a geometric description of the set $E_{2/5}^+(T, \mathbb{P}^n)$ when T is a positive closed current of bidimension $(1, 1)$ on \mathbb{P}^n such that $\|T\| = 1$. The proof of Theorem 3.3 uses ideas from [3, 10, 12] related to self-intersection inequalities for positive closed currents. Using similar ideas, we further show in Theorem 3.4 that if T is a positive closed current of bidimension (p, p) on a compact Kähler manifold (X, ω) , then the set $E_c(T, X)$, $c > 0$, is contained in an analytic set of dimension $\leq p$ whose volume and number of irreducible components are bounded above by a positive constant which depends only on $\|T\|$ and c . In this case, the volumes of analytic subvarieties of X and the mass $\|T\|$ are computed with respect to the fixed Kähler form ω on X .

In Section 4 we study similar geometric properties for upper level sets of positive closed currents of bidimension $(1, 1)$ on multi-projective spaces.

2. PROOFS OF THEOREMS 1.1 AND 1.2

2.1. Proof of Theorem 1.1. Let us start by recalling some terminology. If $A \subset \mathbb{P}^n$ we denote by $\text{Span}(A)$ the smallest linear subspace of \mathbb{P}^n containing A . We say that the points $x_1, \dots, x_{k+1} \in \mathbb{P}^n$, $k \leq n$, are linearly independent if they span a k -dimensional

linear subspace of \mathbb{P}^n . We say that $k > n + 1$ points of \mathbb{P}^n are in general position if any $n + 1$ of them are linearly independent. We will need the following lemma:

Lemma 2.1. *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^n , $0 < p < n$, let $\alpha > 0$, and let V be a $(p + 1)$ -dimensional linear subspace of \mathbb{P}^n . There exists a positive closed current S of bidegree $(1, 1)$ on $V \equiv \mathbb{P}^{p+1}$ such that $\|S\| = \|T\|$ and $E_\alpha^+(T, \mathbb{P}^n) \cap V \subset E_\alpha^+(S, V)$.*

Proof. Assume without loss of generality that $\|T\| = 1$. By [10], there exists a positive closed current T' of bidegree $(1, 1)$ on \mathbb{P}^n such that $\|T'\| = 1$ and $\nu(T', z) = \nu(T, z)$ for every $z \in \mathbb{P}^n$. Demailly's regularization theorem [3, Proposition 3.7] yields a sequence of positive closed currents T'_m of bidegree $(1, 1)$ on \mathbb{P}^n with $\|T'_m\| = 1$, such that each T'_m is smooth outside an analytic subset contained in $E_0^+(T')$ and $\lim_{m \rightarrow \infty} \nu(T'_m, x) = \nu(T', x)$ at each $x \in \mathbb{P}^n$. We write $T'_m = \omega_n + dd^c \varphi_m$ for some ω_n -plurisubharmonic function φ_m on \mathbb{P}^n (see e.g. [9]). Note that by [11], $E_0^+(T') = E_0^+(T)$ is a countable union of analytic subsets of dimension at most p , so $V \setminus E_0^+(T') \neq \emptyset$. Since φ_m is smooth at each point of $V \setminus E_0^+(T')$ the pull-back S_m of T'_m to V , $S_m = \omega_n|_V + dd^c(\varphi_m|_V)$, is a well defined positive closed current of bidegree $(1, 1)$ on V with $\|S_m\| = 1$. By passing to a subsequence, we may assume that S_m converges weakly to a positive closed current S of bidegree $(1, 1)$ on V . Then $\|S\| = 1$ and

$$\nu(S, z) \geq \limsup_{m \rightarrow \infty} \nu(S_m, z) \geq \lim_{m \rightarrow \infty} \nu(T'_m, z) = \nu(T', z),$$

for all $z \in V$. It follows that $E_\alpha^+(T, \mathbb{P}^n) \cap V \subset E_\alpha^+(S, V)$. \square

Proof of Theorem 1.1. Assume for a contradiction that $\dim \text{Span}(E_{(p+1)/(p+2)}^+(T, \mathbb{P}^n)) \geq p + 1$, so there exist linearly independent points $x_1, \dots, x_{p+2} \in E_{(p+1)/(p+2)}^+(T, \mathbb{P}^n)$. Let $V \equiv \mathbb{P}^{p+1}$ be the linear subspace spanned by these points. By Lemma 2.1, there exists a positive closed current S of bidegree $(1, 1)$ on V such that $\|S\| = 1$ and $x_1, \dots, x_{p+2} \in E_{(p+1)/(p+2)}^+(S, V)$. This is in contradiction to [2, Proposition 2.2], which shows that $E_{(p+1)/(p+2)}^+(S, V)$ must be contained in a hyperplane of V . \square

2.2. Proof of Theorem 1.2. We prove first Theorem 1.2 for currents of bidegree $(1, 1)$ on \mathbb{P}^n . This is the contents of Theorem 2.3. Let $\alpha_n = (n - 1)/n$. We begin with the following lemma:

Lemma 2.2. *Let T be a positive closed current of bidegree $(1, 1)$ on \mathbb{P}^n such that $\|T\| = 1$ and $E_{\alpha_n}^+(T, \mathbb{P}^n)$ contains a set $A = \{x_1, \dots, x_{n+1}\}$ of linearly independent points. Then:*

(i) *For every subset $B \subset A$ with $|B| = k + 1$, $k \geq 1$, there exists a positive closed current R_B of bidegree $(1, 1)$ on $\text{Span}(B) \equiv \mathbb{P}^k$ such that $\|R_B\| = 1$ and $E_{\alpha_n}^+(T, \mathbb{P}^n) \cap \text{Span}(B) \subset E_{\alpha_k}^+(R_B, \text{Span}(B))$.*

(ii) *$E_{\alpha_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k \leq n+1} L_{jk}$, where L_{jk} is the line spanned by x_j and x_k .*

Proof. (i) It suffices to prove (i) for $k = n - 1$. Then we apply this inductively to obtain the result for arbitrary k . Assume without loss of generality that $B = \{x_1, \dots, x_n\}$ and let H be the hyperplane spanned by B . Siu's decomposition theorem [11] implies that $T = a[H] + R$, where $[H]$ denotes the current of integration along H , $0 \leq a \leq 1$, and R

is a positive closed current of bidegree $(1, 1)$ on \mathbb{P}^n with generic Lelong number 0 along H . We have

$$1 - a = \|R\| \geq \nu(R, x_{n+1}) = \nu(T, x_{n+1}) > \alpha_n, \quad \text{so } a < 1 - \alpha_n.$$

The current $R' = R/(1 - a)$ has mass $\|R'\| = 1$, generic Lelong number 0 along H , and if $z \in E_{\alpha_n}^+(T, \mathbb{P}^n) \cap H$ then, since $a < 1 - \alpha_n$,

$$(1) \quad \nu(R', z) = \frac{\nu(T, z) - a}{1 - a} > \frac{\alpha_n - a}{1 - a} > \frac{2\alpha_n - 1}{\alpha_n} = \alpha_{n-1}.$$

By [3, Proposition 3.7] there exists a sequence of positive closed currents R'_m of bidegree $(1, 1)$ on \mathbb{P}^n with analytic singularities, such that $\|R'_m\| = 1$, $\nu(R'_m, x) \leq \nu(R', x)$ and $\lim_{m \rightarrow \infty} \nu(R'_m, x) = \nu(R', x)$ for all $x \in \mathbb{P}^n$. It follows that R'_m is smooth at each point of H outside an analytic subset of H , so the pull-back $R'_m|_H$ of R'_m to H is well-defined. Arguing as in the proof of Lemma 2.1 we obtain the current R_B that verifies the desired properties as a weak limit point of $\{R'_m|_H\}$.

(ii) Let H_j denote the hyperplane spanned by $A \setminus \{x_j\}$. We show first that

$$(2) \quad E_{\alpha_n}^+(T, \mathbb{P}^n) \subset \bigcup_{j=1}^{n+1} H_j.$$

Assume that there exists $x_{n+2} \in E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus \bigcup_{j=1}^{n+1} H_j$ and choose $x_0 \in \mathbb{P}^n \setminus \{x_1, \dots, x_{n+2}\}$ so that the points x_0, \dots, x_{n+2} are in general position and $\nu(T, x_0) = 0$. By [3, Proposition 3.7] there exists a positive closed current T' of bidegree $(1, 1)$ on \mathbb{P}^n with analytic singularities, such that $\|T'\| = 1$, $\nu(T', x_j) > \alpha_n$, $j = 1, \dots, n+2$, and T' is smooth near x_0 . Let C be the unique rational normal curve passing through the points x_0, \dots, x_{n+2} (see [7, p. 530]). It follows by [4] and [6] that the measure $T' \wedge [C]$ is well defined, where $[C]$ denotes the current of integration along C . Since C has degree n and using [4, Corollary 5.10], we obtain

$$(3) \quad n = \int_{\mathbb{P}^n} T' \wedge [C] \geq \sum_{j=1}^{n+2} T' \wedge [C](\{x_j\}) \geq \sum_{j=1}^{n+2} \nu(T', x_j) \nu([C], x_j) > (n+2) \alpha_n,$$

a contradiction. This proves (2).

Let now $B_j = A \setminus \{x_j\}$. Applying (2) to the current R_{B_j} given by (i) we obtain

$$E_{\alpha_n}^+(T, \mathbb{P}^n) \cap H_j \subset E_{\alpha_{n-1}}^+(R_{B_j}, H_j) \subset \bigcup_{k=1, k \neq j}^{n+1} \text{Span}(A \setminus \{x_j, x_k\}).$$

Together with (2) this implies that

$$E_{\alpha_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k \leq n+1} \text{Span}(A \setminus \{x_j, x_k\}).$$

Repeating this argument inductively yields (ii). \square

Theorem 2.3. *If T is a positive closed current of bidegree $(1, 1)$ on \mathbb{P}^n with $\|T\| = 1$, then the set $E_{\alpha_n}^+(T, \mathbb{P}^n)$ is either contained a hyperplane or else it is a finite set and $|E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus L| = n - 1$ for some line $L \subset \mathbb{P}^n$.*

Proof. If $E_{\alpha_n}^+(T, \mathbb{P}^n)$ is not contained in a hyperplane then $\dim \text{Span}(E_{\alpha_n}^+(T, \mathbb{P}^n)) = n$ and there exists a set of linearly independent points $A = \{x_1, \dots, x_{n+1}\} \subset E_{\alpha_n}^+(T, \mathbb{P}^n)$. Let L_{jk} denote the line spanned by x_j, x_k .

If $E_{\alpha_n}^+(T, \mathbb{P}^n) = A$ then $|E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus L_{12}| = n - 1$ and we are done. Suppose that there exists $x \in E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus A$. By Lemma 2.2 we have, after relabeling points if necessary, that $x \in L_{12}$.

We show that $E_{\alpha_n}^+(T, \mathbb{P}^n) \subset A \cup L_{12}$. Assume for a contradiction that there exists $y \in E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus (A \cup L_{12})$. Then $y \in L_{jk}$ for some $3 \leq j < k \leq n + 1$. Indeed, if $y \in L_{1k}$ or if $y \in L_{2k}$, $k \geq 3$, then let $B = \{1, 2, k\}$ and R_B be the current on $\text{Span}(B) \equiv \mathbb{P}^2$ provided by Lemma 2.2. Then $\{x, y, x_1, x_2, x_k\} \subset E_{1/2}^+(R_B, \mathbb{P}^2)$, and the set $\{x, y, x_1, x_2, x_k\}$ has at least two points outside each complex line. This is in contradiction to [1, Theorem 1.1]. Hence after relabeling points if necessary we have that $y \in L_{34}$.

Consider now the set $B = \{x_1, x_2, x_3, x_4\}$ and the current $R = R_B$ on $\text{Span}(B) \equiv \mathbb{P}^3$ given by Lemma 2.2, so $\{x, y, x_1, x_2, x_3, x_4\} \subset E_{2/3}^+(R, \mathbb{P}^3)$. If $V_1 = \text{Span}(\{x_2, x_3, x_4\})$, $V_3 = \text{Span}(\{x_1, x_2, x_4\})$, we write, using [11], $R = a[V_1] + b[V_3] + R'$, where R' has generic Lelong number 0 on $V_1 \cup V_3$, $\|R'\| = 1 - a - b$, and

$$\begin{aligned} \nu(R', x_1) &> \frac{2}{3} - b, \quad \nu(R', x_3) > \frac{2}{3} - a, \quad \nu(R', x_j) > \frac{2}{3} - a - b, \quad j = 2, 4, \\ \nu(R', x) &> \frac{2}{3} - b, \quad \nu(R', y) > \frac{2}{3} - a. \end{aligned}$$

Note that $a + b < 1$. By [3, Proposition 3.7] there exists a positive closed current with analytic singularities S of bidegree $(1, 1)$ on \mathbb{P}^3 with $\|S\| = 1 - a - b$ and such that the Lelong numbers of S satisfy the same inequalities as those of R' at the points x, y, x_1, x_2, x_3, x_4 . Moreover, S is smooth at each point where R' has 0 Lelong number. Let C_1 be an irreducible conic in V_1 passing through x_2, x_3, y and a point $w_1 \in V_1$ where $\nu(R', w_1) = 0$. Let C_3 be an irreducible conic in V_3 passing through x_1, x_4, x and a point $w_3 \in V_3$ where $\nu(R', w_3) = 0$. Then the measures $S \wedge [C_j]$, $j = 1, 3$, are well defined and

$$4(1 - a - b) = \int_{\mathbb{P}^3} S \wedge ([C_1] + [C_3]) \geq \nu(S, x) + \nu(S, y) + \sum_{j=1}^4 \nu(S, x_j) > 4 - 4a - 4b,$$

a contradiction.

We conclude that $E_{\alpha_n}^+(T, \mathbb{P}^n) \subset A \cup L_{12}$, hence $|E_{\alpha_n}^+(T, \mathbb{P}^n) \setminus L_{12}| = n - 1$. If $B = \{x_1, x_2, x_3\}$ and R_B is the current on $\text{Span}(B) \equiv \mathbb{P}^2$ given by Lemma 2.2 then $E_{\alpha_n}^+(T, \mathbb{P}^n) \cap L_{12} \subset E_{1/2}^+(R_B, \mathbb{P}^2)$. By [1, Theorem 1.1], the set $E_{1/2}^+(R_B, \mathbb{P}^2)$ is finite since it is not contained in a complex line. It follows that $E_{\alpha_n}^+(T, \mathbb{P}^n)$ is a finite set. \square

Theorem 1.2 for arbitrary p follows at once from Theorem 2.3 and the next proposition.

Proposition 2.4. *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^n , $0 < p < n - 1$, with $\|T\| = 1$ and such that $E_{p/(p+1)}^+(T, \mathbb{P}^n)$ is not contained in a p -dimensional linear subspace of \mathbb{P}^n . If $W = \text{Span}(E_{p/(p+1)}^+(T, \mathbb{P}^n))$ then $\dim W = p + 1$ and there exists a positive closed current R of bidegree $(1, 1)$ on $W \equiv \mathbb{P}^{p+1}$ such that $\|R\| = 1$ and $E_{p/(p+1)}^+(T, \mathbb{P}^n) \subset E_{p/(p+1)}^+(R, W)$.*

Proof. By hypothesis $\dim W \geq p + 1$. Assume for a contradiction that there exist linearly independent points $x_1, \dots, x_{p+3} \in E_{p/(p+1)}^+(T, \mathbb{P}^n)$. Let $U = \text{Span}(\{x_1, \dots, x_{p+2}\})$ and pick $y \in U$ so that the points x_1, \dots, x_{p+2}, y are in general position in $U \equiv \mathbb{P}^{p+1}$. We will construct a positive closed current S of bidegree $(1, 1)$ on U such that $\|S\| = 1$ and $\{x_1, \dots, x_{p+2}, y\} \subset E_{p/(p+1)}^+(S, U)$. By Lemma 2.2 (ii), y must lie in a line spanned by some x_j, x_k , $1 \leq j < k \leq p + 2$. This contradicts the fact that the points x_1, \dots, x_{p+2}, y are in general position in U . The construction of S is as follows. Choose a sequence of points $y_m \in W \setminus U$ such that $y_m \rightarrow y$. Then the points x_1, \dots, x_{p+2}, y_m are linearly independent. Let F_m be an automorphism of \mathbb{P}^n such that $F_m(x_j) = x_j$, $1 \leq j \leq p + 2$, $F_m(x_{p+3}) = y_m$ and set $T_m = (F_m)_* T$. These are positive closed currents of bidimension (p, p) on \mathbb{P}^n with $\|T_m\| = 1$ and $\nu(T_m, x_j) = \nu(T, x_j)$, $1 \leq j \leq p + 2$, $\nu(T_m, y_m) = \nu(T, x_{p+3})$. By passing to a subsequence we may assume that T_m converge weakly to a current T' . Then $\|T'\| = 1$ and by [4],

$$\begin{aligned} \nu(T', x_j) &\geq \limsup_{m \rightarrow \infty} \nu(T_m, x_j) = \nu(T, x_j) > \frac{p}{p+1}, \quad 1 \leq j \leq p+2, \\ \nu(T', y) &\geq \limsup_{m \rightarrow \infty} \nu(T_m, y_m) = \nu(T, x_{p+3}) > \frac{p}{p+1}. \end{aligned}$$

Now Lemma 2.1 applied to T' and U with $\alpha = p/(p+1)$ yields the desired current S .

Hence we have shown that $\dim W = p + 1$. Lemma 2.1 yields a positive closed current R of bidegree $(1, 1)$ on $W \equiv \mathbb{P}^{p+1}$ such that $\|R\| = 1$ and $E_{p/(p+1)}^+(T, \mathbb{P}^n) = E_{p/(p+1)}^+(R, W)$ and the proposition is proved. \square

2.3. Remarks. We start with some examples showing that Theorems 1.1 and 1.2 are sharp. Let $0 < p < n$ and $A = \{x_1, \dots, x_{p+2}\}$ be a set of linearly independent points of \mathbb{P}^n . We set $V_j = \text{Span}(A \setminus \{x_j\})$ and denote by L_{jk} the line spanned by x_j, x_k .

Let $T_1 = \frac{1}{p+2} \sum_{j=1}^{p+2} [V_j]$. Then $\|T_1\| = 1$ and $E_{(p+1)/(p+2)}(T_1, \mathbb{P}^n) = A$ is not contained in a p -dimensional linear subspace of \mathbb{P}^n , so the value $\alpha = (p+1)/(p+2)$ in Theorem 1.1 is sharp.

If $p = 1$ the value $\alpha = 1/2$ in Theorem 1.2 was shown to be sharp in [1]. Assume that $2 \leq p \leq n - 1$, choose points $x \in L_{12} \setminus A$, $y \in L_{34} \setminus A$, and let V_x , resp. V_y , denote the p -dimensional linear subspace of \mathbb{P}^n spanned by $(A \cup \{x\}) \setminus \{x_1, x_2\}$, resp. by $(A \cup \{y\}) \setminus \{x_3, x_4\}$. Note that

$$\begin{aligned} \{x, y\} &\subset V_x \cap V_y \cap V_j \text{ for } j \geq 5, \quad \{x_1, x_2\} \subset V_y \setminus V_x, \quad \{x_3, x_4\} \subset V_x \setminus V_y, \\ x &\in (V_3 \cap V_4) \setminus (V_1 \cup V_2), \quad y \in (V_1 \cap V_2) \setminus (V_3 \cup V_4). \end{aligned}$$

It follows that the bidimension (p, p) current

$$T_2 = \frac{1}{2(p+1)} \left(\sum_{j=1}^4 [V_j] + [V_x] + [V_y] \right) + \frac{1}{p+1} \sum_{j=5}^{p+2} [V_j]$$

has mass $\|T_2\| = 1$ and $\nu(T_2, x) = \nu(T_2, y) = \nu(T_2, x_j) = p/(p+1)$, $1 \leq j \leq p+2$. Thus $E_{p/(p+1)}(T_2, \mathbb{P}^n) \supset A \cup \{x, y\}$, so $|E_{p/(p+1)}(T_2, \mathbb{P}^n) \setminus L| \geq p+1$ for every line $L \subset \mathbb{P}^n$. Hence the value $\alpha = p/(p+1)$ in Theorem 1.2 is sharp.

One can construct a positive closed current T_3 of bidimension (p, p) on \mathbb{P}^n with $\|T_3\| = 1$, for which $E_{p/(p+1)}^+(T_3, \mathbb{P}^n)$ is a countable union of linear subspaces of dimension at most $p-1$ contained in a p -dimensional linear subspace of \mathbb{P}^n . Indeed, let V, V_j , $j \geq 1$, be distinct p -dimensional linear subspaces of \mathbb{P}^n such that $V \cap V_j \neq \emptyset$ for all j , and set

$$T_3 = \frac{p}{p+1} [V] + \frac{1}{p+1} \sum_{j=1}^{\infty} 2^{-j} [V_j].$$

Finally, given any $k \geq 2$, one can construct a positive closed current T_4 of bidimension (p, p) on \mathbb{P}^n such that $\|T_4\| = 1$, $|E_{p/(p+1)}^+(T_4, \mathbb{P}^n) \setminus L| = p$ and $|E_{p/(p+1)}^+(T_4, \mathbb{P}^n) \cap L| = k$, for some line L . Indeed, pick distinct points $y_j \in L_{12} \setminus \{x_1, x_2\}$, $1 \leq j \leq k-2$, and let $W_j = \text{Span}(\{y_j, x_3, \dots, x_{p+2}\})$. If $0 < \varepsilon < \frac{1}{k-1}$ let

$$T_4 = \frac{p-\varepsilon}{p(p+1)} \sum_{j=3}^{p+2} [V_j] + \frac{1+\varepsilon}{k(p+1)} \left([V_1] + [V_2] + \sum_{j=1}^{k-2} [W_j] \right).$$

Then $\|T_4\| = 1$ and

$$\nu(T_4, x_1) = \nu(T_4, x_2) = \nu(T_4, y_j) = \frac{p-\varepsilon}{p+1} + \frac{1+\varepsilon}{k(p+1)} = \frac{p}{p+1} + \frac{1-(k-1)\varepsilon}{k(p+1)} > \frac{p}{p+1},$$

$$\nu(T_4, x_j) = \frac{(p-1)(p-\varepsilon)}{p(p+1)} + \frac{1+\varepsilon}{p+1} = \frac{p}{p+1} + \frac{\varepsilon}{p(p+1)} > \frac{p}{p+1}, \quad j \geq 3.$$

Hence T_4 satisfies the desired properties with $L = L_{12}$.

We conclude this section by showing the following property of the set $E_{p/(p+1)}^+(T, \mathbb{P}^n)$:

Proposition 2.5. *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^n , $0 < p < n$, with $\|T\| = 1$, such that the set $E_{p/(p+1)}^+(T, \mathbb{P}^n)$ contains the linearly independent points x_1, \dots, x_{p+1} . If $V = \text{Span}(\{x_1, \dots, x_{p+1}\})$ and c is the generic Lelong number of T along V then $c > 0$.*

Proof. Assume that $c = 0$. Applying [10] and [3, Proposition 3.7] as in the proof of Lemma 2.1 we obtain a positive closed current S of bidegree $(1, 1)$ on \mathbb{P}^n , with analytic singularities, such that $\|S\| = 1$, $\nu(S, x_j) > p/(p+1)$ for $1 \leq j \leq p+1$, and S is smooth at each point of $V \equiv \mathbb{P}^p$ outside an analytic subset of V . Then the pull-back R of S to V is well defined, it has unit mass and Lelong number $> p/(p+1)$ at the linearly independent points x_j . This contradicts Theorem 1.1 (or [2, Proposition 2.2]). \square

3. PROOF OF THEOREM 1.3

We prove first Theorem 1.3 for currents of bidegree $(1, 1)$ on \mathbb{P}^n , $n \geq 3$. This is done in the following lemma. Let $\beta_n = (3n - 4)/(3n - 1)$.

Lemma 3.1. *Let T be a positive closed current of bidegree $(1, 1)$ on \mathbb{P}^n , $n \geq 3$, such that $\|T\| = 1$ and $E_{\beta_n}^+(T, \mathbb{P}^n)$ contains a set $A = \{x_1, \dots, x_{n+1}\}$ of linearly independent points. Then:*

(i) *For every subset $B \subset A$ with $|B| = k + 1$, $k \geq 2$, there exists a positive closed current R_B of bidegree $(1, 1)$ on $\text{Span}(B) \equiv \mathbb{P}^k$ such that $\|R_B\| = 1$ and $E_{\beta_n}^+(T, \mathbb{P}^n) \cap \text{Span}(B) \subset E_{\beta_k}^+(R_B, \text{Span}(B))$.*

(ii) *$E_{\beta_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k < l \leq n+1} P_{jkl}$, where $P_{jkl} = \text{Span}(\{x_j, x_k, x_l\})$.*

(iii) *There exist conics $C_{jkl} \subset P_{jkl}$ and points $z_{jkl} \in P_{jkl}$ such that $E_{\beta_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k < l \leq n+1} (C_{jkl} \cup \{z_{jkl}\})$.*

Proof. Assertions (i) and (ii) are shown exactly as in the proof of Lemma 2.2, using in (1) the fact that $(2\beta_n - 1)/\beta_n = \beta_{n-1}$, and in (3) the fact that $n > (n + 2)\beta_n$ implies $n \leq 2$, which contradicts the assumption that $n \geq 3$.

(iii) Let $B = \{x_j, x_k, x_l\}$, where $1 \leq j < k < l \leq n + 1$. By (i) there exists a positive closed current R of bidegree $(1, 1)$ on P_{jkl} such that $\|R\| = 1$ and $E_{\beta_n}^+(T, \mathbb{P}^n) \cap P_{jkl} \subset E_{2/5}^+(R, P_{jkl})$. Theorem 1.2 in [1] shows that there exist a conic $C_{jkl} \subset P_{jkl}$ and a point $z_{jkl} \in P_{jkl}$ such that $E_{2/5}^+(R, P_{jkl}) \subset C_{jkl} \cup \{z_{jkl}\}$. Hence (iii) follows from (ii). \square

The next proposition is proved exactly like Proposition 2.4, by using Lemma 3.1 (ii).

Proposition 3.2. *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^n such that $1 < p < n - 1$, $\|T\| = 1$, and the set $E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n)$ is not contained in a p -dimensional linear subspace of \mathbb{P}^n . If $W = \text{Span}(E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n))$ then $\dim W = p + 1$ and there exists a positive closed current R of bidegree $(1, 1)$ on $W \equiv \mathbb{P}^{p+1}$ such that $\|R\| = 1$ and $E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n) \subset E_{(3p-1)/(3p+2)}^+(R, W)$.*

Theorem 1.3 for arbitrary $p \geq 2$ now follows at once from Proposition 3.2 and from Lemma 3.1 (iii).

We now turn our attention to the case of currents of bidimension $(1, 1)$. If L_1, L_2 are non-concurrent lines in \mathbb{P}^n and $T = ([L_1] + [L_2])/2$ then $\dim \text{Span}(E_{2/5}^+(T)) = 3$, and Theorem 1.3 does not hold for $p = 1$. However, we have the following geometric property of the set $E_{2/5}^+(T)$ in this setting:

Theorem 3.3. *Let T be a positive closed current of bidimension $(1, 1)$ on \mathbb{P}^n with $\|T\| = 1$. If $|E_{2/5}^+(T)| > 37$ then there exists a curve $C \subset \mathbb{P}^n$ of degree at most 2 such that $|E_{2/5}^+(T) \setminus C| \leq 1$.*

Proof. We consider two mutually exclusive cases.

Case 1: The set $E_{\gamma}^+(T)$ is infinite for some $\gamma > 1/3$. Then, by [11], $E_{\gamma}(T)$ contains an irreducible curve X and $T = T' + \gamma[X]$, where T' is a positive closed current and

$\deg X \leq 1/\gamma < 3$. If $\deg X = 2$ then, by [5, Proposition 0], X is an irreducible plane conic. Moreover, $\|T'\| = 1 - 2\gamma < 1/3$, so $E_{2/5}^+(T) \subset X$. If $\deg X = 1$ then $\|T'\| = 1 - \gamma < 2/3$. It follows by [1, Theorem 1.1] that $|E_{1/3}(T') \setminus L| \leq 1$ for some line L . Since $E_{2/5}^+(T) \subset X \cup E_{1/3}(T')$ we conclude that $|E_{2/5}^+(T) \setminus C| \leq 1$, where $C = X \cup L$.

Case 2: The set $E_\gamma^+(T)$ is finite for all $\gamma > 1/3$. By [10], there is a positive closed current S of bidegree $(1, 1)$ on \mathbb{P}^n such that $\|S\| = \|T\| = 1$, and S has the same Lelong number as T at every point. Fix $\gamma \in (1/3, 2/5)$. By Demailly's regularization theorem applied to S (Main Theorem 1.1 in [3], where we can take $u = 0$ since we work on \mathbb{P}^n), for any $\epsilon > 0$ there is a positive closed current $S_{\epsilon, \gamma}$ of bidegree $(1, 1)$ on \mathbb{P}^n with the following properties:

(i) $S_{\epsilon, \gamma}$ is smooth on $\mathbb{P}^n \setminus E_\gamma(T)$, hence $S_{\epsilon, \gamma}$ is smooth outside a finite set.

(ii) $\|S_{\epsilon, \gamma}\| = 1 + \epsilon$ and $\nu(S_{\epsilon, \gamma}, x) = \max\{\nu(T, x) - \gamma, 0\}$ at each $x \in \mathbb{P}^n$.

Let $A = E_{2/5}^+(T)$. Then $\nu(S_{\epsilon, \gamma}, x) > 2/5 - \gamma$ for $x \in A$. Since $S_{\epsilon, \gamma}$ is smooth outside a finite set the measure $S_{\epsilon, \gamma} \wedge T$ is well defined [4]. We estimate $|A|$ as follows:

$$1 + \epsilon = \int_{\mathbb{P}^n} S_{\epsilon, \gamma} \wedge T \geq \sum_{x \in A} \nu(S_{\epsilon, \gamma}, x) \nu(T, x) > \frac{2}{5} \left(\frac{2}{5} - \gamma \right) |A|.$$

Choosing $\epsilon > 0$ very small and $\gamma > 1/3$ very close to $1/3$ we find that $|A| \leq 37$. \square

The argument in Case 2 of the proof of Theorem 3.3 can be used to prove a more general result.

Theorem 3.4. *Let (X, ω) be a compact Kähler manifold of dimension n , and T be a positive closed current of bidimension (p, p) on X . For any $c > 0$, the set $E_c(T)$ is contained in an analytic set of dimension $\leq p$ whose volume and number of irreducible components are bounded above by a constant $K(\|T\|, c)$ depending only on $\|T\|$ and c .*

Proof. Recall that $\|T\| = \int_X T \wedge \omega^p$ and if Z is an analytic subvariety of X then $\text{vol } Z = \sum_V \int_V \omega^{\dim V}$, where the sum is over all irreducible components V of Z . By Lelong's theorem, there is a positive number μ_0 such that any subvariety of X has volume at least μ_0 . Therefore the number of irreducible components of Z is $\leq (\text{vol } Z)/\mu_0$.

The proof is by induction on p . If $p = 0$ then T is a measure and $E_c(T)$ is a finite set whose cardinality is $\leq \|T\|/c$.

Assume that the theorem is true for $p = p_0$. We need to prove it for $p = p_0 + 1$. Let us define $A_{c/2, p_0+1}(T)$ to be the union of all irreducible components of dimension $p_0 + 1$ of the analytic set $E_{c/2}(T)$. Set

$$T' = T - \sum_{V \subset A_{c/2, p_0+1}(T)} \lambda_V(T)[V],$$

where the sum is over all irreducible components V of $A_{c/2, p_0+1}(T)$ and $\lambda_V(T)$ is the generic Lelong number of T along V . By [11] T' is a positive closed current of bidimension (p_0+1, p_0+1) and $\|T'\| \leq \|T\|$. Moreover the set $E_{c/2}(T')$ has dimension at most p_0 , since

$E_{c/2}(T') \subset E_{c/2}(T)$ and T' does not charge any irreducible component V of $A_{c/2, p_0+1}(T)$. Since $\lambda_V(T) \geq c/2$, we have that

$$\text{vol } A_{c/2, p_0+1}(T) = \left\| \sum_{V \subset A_{c/2, p_0+1}(T)} [V] \right\| \leq 2\|T\|/c.$$

By [12, Theorem 3.1] there is a positive closed current R of bidegree $(1, 1)$ on X which has the same Lelong number as T' at every point and such that $\|R\| \leq C_1\|T'\|$, where $C_1 > 0$ is a constant depending only on X and ω . By Demailly's regularization theorem applied to R (Main Theorem 1.1 in [3]), there is a positive closed current R' of bidegree $(1, 1)$ on X such that: $\|R'\| \leq C_2\|R\|$, where C_2 is a constant depending only on X and ω , R' is smooth on $X \setminus E_{c/2}(T')$, and $\nu(R', x) = \max\{\nu(T', x) - c/2, 0\}$ for every $x \in X$. Since $\dim E_{c/2}(T') \leq p_0$, $T_1 = R' \wedge T'$ is a well defined positive closed current of bidimension (p_0, p_0) by [4]. Moreover, $\|T_1\| \leq C_3\|T'\|\|R'\| \leq C\|T\|^2$, where $C = C_1C_2C_3$ and C_3 is a constant depending only on X and ω . By Demailly's comparison theorem for Lelong numbers [4] we have for $x \in E_c(T) \setminus A_{c/2, p_0+1}(T)$,

$$\nu(T_1, x) \geq \nu(R', x)\nu(T', x) \geq c^2/2.$$

Therefore, if W is the union of all the irreducible components of $E_c(T)$ that are not contained in $A_{c/2, p_0+1}(T)$, then $W \subset E_{c^2/2}(T_1)$. The induction assumption implies that $E_{c^2/2}(T_1)$ is contained in an analytic subset of dimension $\leq p_0$ whose volume is $\leq K(C\|T\|^2, c^2/2)$. Thus the proof for the case $p = p_0 + 1$ is complete. \square

We note that it is not true that the number of irreducible components of the set $E_c(T)$ itself is bounded by a constant depending only on $\|T\|$ and c , as the following simple example shows. Let L_j , $0 \leq j \leq k+1$, be lines in \mathbb{P}^n so that $L_0 \cap L_j = \{z_j\}$, $j \geq 1$, and no three of them pass through the same point. Let

$$T = \left(\frac{1}{2} - \frac{1}{2k} \right) [L_0] + \frac{1}{2k} \sum_{j=1}^{k+1} [L_j].$$

Then $\|T\| = 1$ and $E_{1/2}(T, \mathbb{P}^n) = \{z_1, \dots, z_{k+1}\}$, provided that $k \geq 3$.

4. POSITIVE CLOSED CURRENTS ON $\mathbb{P}^m \times \mathbb{P}^n$

We prove here certain geometric properties of the upper level sets of Lelong numbers of positive closed currents of bidimension $(1, 1)$ on a multiprojective space

$$X = \mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}_z^m \times \mathbb{P}_w^n.$$

Let $\pi_z : X \longrightarrow \mathbb{P}_z^m$, $\pi_w : X \longrightarrow \mathbb{P}_w^n$, denote the canonical projections and

$$z = [z_0 : \dots : z_m], \quad w = [w_0 : \dots : w_n],$$

denote the homogeneous coordinates on \mathbb{P}^m , respectively on \mathbb{P}^n . Set

$$\omega_z = \pi_z^* \omega_m, \quad \omega_w = \pi_w^* \omega_n,$$

where ω_m and ω_n are the Fubini-Study forms on \mathbb{P}^m , respectively \mathbb{P}^n . The Dolbeault cohomology group $H^{m+n-1, m+n-1}(X, \mathbb{R})$ is generated by the forms $\omega_z^m \wedge \omega_w^{n-1}$ and $\omega_z^{m-1} \wedge \omega_w^n$. Let

$$\theta_{a,b} = a\omega_z^m \wedge \omega_w^{n-1} + b\omega_z^{m-1} \wedge \omega_w^n, \quad a, b \geq 0,$$

and let $\mathcal{T}_{a,b}$ denote the space of positive closed currents of bidimension $(1, 1)$ on X which lie in the cohomology class of $\theta_{a,b}$.

Proposition 4.1. *If $T \in \mathcal{T}_{a,b}$ then $E_{(a+b)/2}^+(T, X) \subset \pi_z^{-1}(x)$ for some $x \in \mathbb{P}^m$ or $E_{(a+b)/2}^+(T, X) \subset \pi_w^{-1}(y)$ for some $y \in \mathbb{P}^n$.*

Proof. We may assume that $a + b = 1$ and that for any $x \in \mathbb{P}^m$ we have $E_{1/2}^+(T, X) \not\subset \pi_z^{-1}(x)$. Then there exist points $p_1, p_2 \in E_{1/2}^+(T, X)$ such that $\pi_z(p_1) \neq \pi_z(p_2)$.

We claim that $\pi_w(p_1) = \pi_w(p_2)$. Indeed, suppose $\pi_w(p_1) \neq \pi_w(p_2)$. Composing with an automorphism of X we may assume that

$$p_1 = ([1 : 0 : \dots : 0], [1 : 0 : \dots : 0]), \quad p_2 = ([0 : \dots : 0 : 1], [0 : \dots : 0 : 1]).$$

Consider the function on $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$,

$$u(z_0, \dots, z_m, w_0, \dots, w_n) = \max \left\{ \left(\sum_{j=1}^m |z_j|^2 \right) \left(\sum_{k=0}^{n-1} |w_k|^2 \right), \left(\sum_{j=0}^{m-1} |z_j|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) \right\}.$$

The current $\frac{1}{2} dd^c \log u$ determines a positive closed current S of bidegree $(1, 1)$ on X in the cohomology class of $\omega_z + \omega_w$ (see e.g. [8, 2]). Note that S has bounded local plurisubharmonic potentials on $X \setminus \{p_1, p_2\}$ and $\nu(S, p_1) = \nu(S, p_2) = 1$. Then

$$a + b = \int_X T \wedge S \geq T \wedge S(\{p_1\}) + T \wedge S(\{p_2\}) \geq \nu(T, p_1) + \nu(T, p_2) > 1,$$

a contradiction.

Hence $\pi_w(p_1) = \pi_w(p_2) = y$. We show that $E_{1/2}^+(T, X) \subset \pi_w^{-1}(y)$. If not, there exists $p \in E_{1/2}^+(T, X)$ with $\pi_w(p) \neq y$. Since $\pi_z(p_1) \neq \pi_z(p_2)$ we may assume that $\pi_z(p) \neq \pi_z(p_1)$. Then we obtain a contradiction as above, working with the points p, p_1 instead of the points p_1, p_2 . \square

Our next result is in analogy to that of Theorem 1.1. By *vertical line in X* we mean a line $L \subset \pi_z^{-1}(x) \equiv \mathbb{P}^n$ for some $x \in \mathbb{P}^m$, while by *horizontal line in X* we mean a line $L \subset \pi_w^{-1}(y) \equiv \mathbb{P}^m$ for some $y \in \mathbb{P}^n$.

Proposition 4.2. *Let $T \in \mathcal{T}_{a,b}$ and $\alpha = \max \left\{ \frac{2a+b}{3}, \frac{a+2b}{3} \right\}$. Then $E_\alpha^+(T, X)$ is contained in a vertical line in X or in a horizontal line in X .*

Proof. Since $\alpha \geq (a+b)/2$ it follows by Proposition 4.1 that $E_\alpha^+(T, X) \subset \pi_z^{-1}(x)$ for some $x \in \mathbb{P}^m$ or $E_\alpha^+(T, X) \subset \pi_w^{-1}(y)$ for some $y \in \mathbb{P}^n$. Without loss of generality we may assume that $E_\alpha^+(T, X) \subset \pi_z^{-1}(x)$, where $x = [1 : 0 : \dots : 0] \in \mathbb{P}^m$. We will show that $E_\alpha^+(T, X)$ is contained in a line in $\pi_z^{-1}(x) \equiv \mathbb{P}^n$.

Suppose for a contradiction that there exist non-collinear points $y^1, y^2, y^3 \in \mathbb{P}^n$ such that $p_j := (x, y^j) \in E_\alpha^+(T, X)$. We can find homogeneous quadratic polynomials

P_1, \dots, P_n on \mathbb{C}^{n+1} such that the set $\{P_1 = 0\} \cap \dots \cap \{P_n = 0\} \subset \mathbb{P}^n$ is finite and it contains the points y^1, y^2, y^3 . Moreover, the $2\omega_n$ -plurisubharmonic function on \mathbb{P}^n determined by $\frac{1}{2} \log \left(\sum_{k=1}^n |P_k|^2 \right)$ has Lelong number 1 at each y^j . Consider the function on $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$,

$$u(z, w) = \max \left\{ \left(\sum_{j=0}^m |z_j|^2 \right) \left(\sum_{k=1}^n |P_k(w)|^2 \right), \left(\sum_{j=1}^m |z_j|^2 \right) \left(\sum_{k=0}^n |w_k|^2 \right)^2 \right\}.$$

The current $\frac{1}{2} dd^c \log u$ determines a positive closed current S of bidegree $(1, 1)$ on X in the cohomology class of $\omega_z + 2\omega_w$ (see e.g. [8, 2]). Note that S has bounded local plurisubharmonic potentials on the complement of a finite subset of X and $\nu(S, p_j) = 1$. Then

$$2a + b = \int_X T \wedge S \geq \sum_{j=1}^3 T \wedge S(\{p_j\}) \geq \sum_{j=1}^3 \nu(T, p_j) > 3\alpha,$$

a contradiction. This completes the proof. \square

We end this section with some examples showing that Propositions 4.1 and 4.2 are sharp. Consider distinct points $x_1, x_2 \in \mathbb{P}^m$, $y_1, y_2 \in \mathbb{P}^n$, and let $p_{jk} = (x_j, y_k) \in X$. For $j = 1, 2$, denote by $V_j \subset \pi_z^{-1}(x_j)$ the vertical line determined by the points p_{j1}, p_{j2} , and by $H_j \subset \pi_w^{-1}(y_j)$ the horizontal line determined by the points p_{1j}, p_{2j} . Let $a, b > 0$ and $T_1, T_2 \in \mathcal{T}_{a,b}$ be the currents

$$T_1 = \frac{a}{2} ([V_1] + [V_2]) + b[H_1], \quad T_2 = a[V_1] + \frac{b}{2} ([H_1] + [H_2]).$$

Then

$$\begin{aligned} \{p_{11}, p_{21}\} &\subset E_{(a+b)/2}^+(T_1, X) \subset H_1 \subset \pi_w^{-1}(y_1), \\ \{p_{11}, p_{12}\} &\subset E_{(a+b)/2}^+(T_2, X) \subset V_1 \subset \pi_z^{-1}(x_1). \end{aligned}$$

Hence the set $E_{(a+b)/2}^+(T, X)$ can be contained in a vertical fiber or in a horizontal fiber, regardless of how a compares to b . Assume next that $a \geq b$ and let

$$T_3 = \frac{a+b}{2} [V_1] + \frac{a-b}{2} [V_2] + b[H_1], \quad \text{so } V_1 \cup \{p_{21}\} \subset E_{(a+b)/2}(T_3, X).$$

Hence $E_{(a+b)/2}(T_3, X) \not\subset \pi_z^{-1}(x)$ for any $x \in \mathbb{P}^m$, $E_{(a+b)/2}(T_3, X) \not\subset \pi_w^{-1}(y)$ for any $y \in \mathbb{P}^n$, and Proposition 4.1 is sharp.

The next example shows the sharpness of Proposition 4.2. Assume $a \geq b$, let $x \in \mathbb{P}^m$, let $y_1, y_2, y_3 \in \mathbb{P}^n$ be non-collinear points, and set $p_j = (x, y_j)$. Denote by $V_{jk} \subset \pi_z^{-1}(x)$ the vertical line determined by the points p_j, p_k , and by $H_j \in \pi_w^{-1}(y_j)$ a horizontal line containing p_j . Then

$$T_4 = \frac{a}{3} ([V_{12}] + [V_{23}] + [V_{13}]) + \frac{b}{3} ([H_1] + [H_2] + [H_3]) \in \mathcal{T}_{a,b}$$

has $\nu(T_4, p_j) = (2a + b)/3 = \alpha$, where α is as in Proposition 4.2. Thus $E_\alpha(T_4, X)$ is not contained in a vertical line or in a horizontal line in X .

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DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244-1150, USA
E-mail address: dcoman@syr.edu

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244-1150, USA
E-mail address: tutruong@syr.edu